On Some Aspects of Linear Connections in Noncommutative Geometry

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Abstract

We discuss two concepts of metric and linear connections in noncommutative geometry, applying them to the case of the product of continuous and discrete (two-point) geometry.

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1 Introduction

Noncommutative geometry [1-5] is one of the most attractive mathematical concepts in physics that could be applied in fundamental field theory. So far, the investigations of gravity in this framework have been concentrated on the case of the product of Minkowski space by a two-point space, which has been motivated by the Standard Model (see [6-8])
Their methods, however, did not use the whole structure of noncommutative geometry, in particular, the definitions of metric and linear connections did not use the bimodule structure of differential forms.
Only recently some general ideas concerning linear connection and metric have been proposed and discussed for other examples . They [9-11] are based on the idea that a key role in the introduction of these structure plays as generalised permutation operation.
A different model of the generalisation of the metric as well as a simple model of gravity on the product of Minkowski space and two-point space has been already discussed by us earlier, with some encouraging results [12].
In this paper, we shall discuss two methods of construction of the metric and linear connections based on two different concepts, first as proposed in [9-11], based on symme tric metric and bimodule property of linear connection, the other one, which uses hermitian metric and left-linearity of linear connection and follows the idea of our previous paper (though it differs in few significant points). We shall try to derive the consequences of these models for the considered example. Our main aim is to determine what conditions are necessary, what could be abandoned and what are too strict for noncommutative geometry. Of course, the basic test is the agreement with the standard differential geometry.

2 Notation

Our basic data is a (graded) differential algebra Ω with the external derivative d obeying the graded Leibniz rule:

$$d(u \wedge v) = du \wedge v + (-1)^{\deg u} u \wedge dv, \tag{1}$$

We shall denote by Ω^n a bimodule of n-forms, $n \geq 1$ and we shall write $\mathcal{A}$ for Ω^0.
Let π_n be the canonical projection $\pi_n : \Omega^{\otimes n} \to \Omega^n$, $n \geq 2$, for simplicity we shall often write π unless it is necessary to specify the index n. We assume

also that our external algebra is a graded $\star$-algebra and we have

$$d(\omega^\star) \;=\; (d\omega)^\star, \tag{2}$$

To end this section let us remind the basic notation of an example of a noncommutative differential calculus on a product of $\mathbb{R}^n$ and a two-point space.

The algebra $\mathcal{A}$ contains of functions on $R^n \times \mathbb{Z}_2$, with pointwise addition and multiplication (also with respect to the discrete coordinates). The bimodule of one-forms is generated by $n+1$ elements: $\{dx^i\}_{i=1,\ldots,n}$ and χ, with the following set of multiplication properties:

$$f(x,p)dx^i \;=\; dx^i f(x,p) \tag{3}$$

$$f(x,p)\chi \;=\; \chi f(x,-p) \tag{4}$$

where p denotes the discrete coordinate taking values $+$ and $-$.

The external derivative is defined as follows:

$$df(x,p) \;=\; \sum_{i=1}^{n} dx^i \partial_i f(x,p) + \chi \partial f(x,p) \tag{5}$$

where ∂_i is the usual partial derivative and $\partial f = (1-\mathcal{R})f$, $\mathcal{R}$ being the morphism, which flips the discrete coordinate: $\mathcal{R}f(x,p) = f(x,-p)$.

The external algebra is built with the following multiplication rules:

$$dx^i \wedge dx^j \;=\; -dx^j \wedge dx^i, \tag{6}$$

$$dx^i \wedge \chi \;=\; -\chi \wedge dx^i, \tag{7}$$

$$d(\chi) \;=\; 2\chi \wedge \chi, \tag{8}$$

and is infinite-dimensional, as $\chi \wedge \chi$ does not vanish. One can introduce a $\star$-algebra structure on this algebra, assuming that:

$$(dx^i)^\star \;=\; dx^i, \qquad \chi^\star = -\chi. \tag{9}$$

The differential calculus constructed in the above described way is just a tensor product of external algebras on the continuous space (which is a standard one) and the discrete two-point space (which is an universal differential calculus).

3 Symmetrization and antisymmetrization

In the classical differential geometry the external algebra is defined as an antisymmetrization of the tensor algebra of one-forms, therefore these operations precede the construction of differential calculus. In noncommutative geometry this situation could be different and we may choose between several possibilities, all of them coinciding in the case of commutative differential structures.

3.1 Antisymmetrization

We may choose a similar way as in the standard differential geometry and, having constructed the first order differential calculus, (i.e. bimodule Ω^1 and $d : \mathcal{A} \to \Omega^1$, which obeys the Leibniz rule) we may look for a bimodule isomorphism σ:

$$\sigma : \Omega^1 \otimes_{\mathcal{A}} \Omega^1 \to \Omega^1 \otimes_{\mathcal{A}} \Omega^1, \tag{10}$$

which would correspond to the permutation: $dx^a \otimes_{\mathcal{A}} dx^b \to dx^b \otimes_{\mathcal{A}} dx^a$.
Then, we define the noncommutative analogue of the symmetrizing morphism on $\Omega^1 \otimes_{\mathcal{A}} \Omega^1$ as $1 - \sigma$ and, consequently, the bimodule of two-forms as a quotient bimodule $\Omega^1 \otimes_{\mathcal{A}} \Omega^1 / S$, where $S = \text{Ker } (1 - \sigma)$. However, we must ensure that the following consistency conditions hold: for any elements $a_i, b_i \in \mathcal{A}$ we must have:

$$\sum_i a_i db_i = 0 \ \Rightarrow \sum_i da_i \otimes_{\mathcal{A}} db_i \in S. \tag{11}$$

If σ satisfies a braid group relation on $\Omega^{\otimes_{\mathcal{A}} 3}$ then the construction of the whole differential algebra follows directly, let us stress that it is not necessary to require $\sigma^2 = 1$.

3.2 Symmetrization

Having defined σ and the external algebra we might define a symmetrization morphism on $\Omega^1 \otimes_{\mathcal{A}} \Omega^1$ as $1 + \sigma$, however, we cannot guarantee without some additional assumptions that:

$$\pi \circ (1 + \sigma) = 0, \tag{12}$$

Indeed, since Ker $\pi =$ Ker $(1 - \sigma)$, if $(1 + \sigma)\xi \in$ Ker π we would have that either $\sigma\xi = -\xi$ or $(1 - \sigma)(1 + \sigma)\xi = 0$, so that $\sigma^2\xi = \xi$ in both cases.

Therefore, $\sigma^2 = 1$ is a necessary requirement (it is obvious that it is also sufficient) for (12) to hold.
Another option (which has been discussed by [9]) is to assume the existence of σ and the (23) relation without deriving the external calculus from σ, in that case, however, we can lose strict relations between the calculus and σ, and the choice of σ could be rather ambiguous.

3.3 Symmetrization and Antisymetrization - All In One

In what follows we shall discuss a possibility of deriving the symmetrization and antisymmetrization operations from the external algebra itself. Of course, without some additional assumptions this is not possible, however, as one could see that these assumptions are rather natural, we shall present the idea here.
Let S denote Ker π and j be the inclusion of S in $\Omega^1 \otimes_{\mathcal{A}} \Omega^1$. Then the following is a short exact sequence of bimodules over $\mathcal{A}$:

$$0 \hookrightarrow S \overset{j}{\hookrightarrow} \Omega^1 \otimes_{\mathcal{A}} \Omega^1 \overset{\pi}{\rightarrow} \Omega^2 \rightarrow 0. \tag{13}$$

If Ω^2 is a projective module the above exact sequence is a split sequence, i.e. there exist maps r and ρ:

$$r: \quad \Omega^1 \otimes_{\mathcal{A}} \Omega^1 \rightarrow S \quad r \circ j = \mathrm{id}_S \tag{14}$$

$$\rho: \quad \Omega^2 \rightarrow \Omega^1 \otimes_{\mathcal{A}} \Omega^1 \quad \pi \circ \rho = \mathrm{id}_{\Omega^1 \otimes_{\mathcal{A}} \Omega^1} \tag{15}$$

and, moreover:

$$\Omega^1 \otimes_{\mathcal{A}} \Omega^1 \cong S \oplus \Omega^2. \tag{16}$$

The latter allows us to introduce a natural symmetrization and antisymmetrization operations on $\Omega^1 \otimes_{\mathcal{A}} \Omega^1$. For every $\xi \in \Omega^1 \otimes_{\mathcal{A}} \Omega^1$ we can represent it as a sum $\xi_s + \xi_a$, where $\xi_s \in j(S)$ and $\xi_a \in \rho(\Omega^2)$. Then the following map:

$$\sigma: \quad \sigma(\xi) = \xi_s - \xi_a, \tag{17}$$

is a bimodule homomorphism such that Ker $\sigma = 0$ and $\sigma^2 = 1$. One can easily verify that $1-\sigma$ is then a projection on $\rho(\Omega^2)$ and $1+\sigma$ is a projection on $j(S)$.

3.4 Example

In the example discussed in this paper the situation is rather simple, as the only nontrivial noncommutative part comes from the discrete geometry. As all bimodules Ω^n are free, we may use results of the last section. We shall write here only the resulting homomorphism σ:

$$\sigma(dx^i \otimes_{\mathcal{A}} dx^j) = dx^j \otimes_{\mathcal{A}} dx^i \tag{18}$$

$$\sigma(dx^i \otimes_{\mathcal{A}} \chi) = \chi \otimes_{\mathcal{A}} dx^i \tag{19}$$

$$\sigma(\chi \otimes_{\mathcal{A}} \chi) = -\chi \otimes_{\mathcal{A}} \chi \tag{20}$$

4 Metric

The construction of metric is one of the most important issues in noncommutative geometry. First, it is required in the studies of field theories (in particular gauge theories) in this framework, secondly it is a crucial step towards the analysis of gravity. We shall outline here the commonly used definition and discuss several points, which are still not well established.

4.1 Definition

It has been almost generally agreed that the proper generalisation of the metric tensor is a bimodule map:

$$g: \quad \Omega^1 \otimes_{\mathcal{A}} \Omega^1 \ \to \mathcal{A}, \tag{21}$$

as it is a natural extension of the standard bilinear map to the noncommutative situation. If our differential algebra is a $\star$-algebra one should also postulate:

$$g(u^\star, v^\star) \ = \ g(v, u)^\star, \tag{22}$$

which guarantees that $g(u, u^\star)$ is self-adjoined.
The above mentioned properties of the metric tensor translate easily from the standard differential geometry into the noncommutative geometry, however, the problems start, when we begin to analyse other features of the metric tensor.

4.2 Symmetry

In the standard differential geometry one postulates that the metric is symmetric, i.e. $g(u,v) = g(v,u)$ for any one-forms u, v. Of course, this requirement cannot hold in noncommutative geometry, however, one could think of replacing it by a different one, which recover this property in the commutative limit.
The ambiguity comes from the fact that even in classical geometry one may look at this property of the metric from two different points. First, one may view the symmetry as related to the hermitian metric condition, then the appropriate generalisation should be just (22). Another point of view relates the symmetry of the metric to the symmetrization operation on the bimodule $\Omega^1 \otimes_{\mathcal{A}} \Omega^1$, then the corresponding generalisation should take the form [9]:

$$g \circ \sigma \;=\; g, \tag{23}$$

where σ is the bimodule isomorphism discussed earlier. We shall now investigate the consequences of each of these definitions in our example.

4.2.1 Symmetric metric - example

From the definition (21) we immediately get that the metric evaluated on the generating one forms must be:

$$\begin{aligned} g(dx^i \otimes_{\mathcal{A}} dx^j) &= g^{ij} && (24)\\ g(dx^i \otimes_{\mathcal{A}} \chi) &= 0 && (25)\\ g(\chi \otimes_{\mathcal{A}} dx^i) &= 0 && (26)\\ g(\chi \otimes_{\mathcal{A}} chi) &= g && (27) \end{aligned}$$

so that the 'mixed' components must vanish and g^{ij},g denote the nonzero elements of the algebra $\mathcal{A}$.
The hermitian metric condition (22) together with (9) relations gives us:

$$\begin{aligned} (g^{ij})^\star &= g^{ji}, && (28)\\ g^\star &= g. && (29) \end{aligned}$$

If we require the additional symmetry (23), we obtain:

$$\begin{aligned} g^{ij} &= g^{ji}, && (30)\\ g &= -g. && (31) \end{aligned}$$

so that g^{ij} is a real and symmetric tensor and g vanishes. The latter property is rather inconvenient and we shall now generalise it and discuss in details.

4.3 Metric on Universal Differential Calculus

So far, we have encountered a problem with the existence of a (nontrivial) metric on the discrete space of two-points, if we assumed its symmetry (23). This feature appears every time we have an universal differential calculus:

Observation 1

If Ω is an universal differential calculus, then there exists no non-trivial metric on $\Omega^1 \otimes_{\mathcal{A}} \Omega^1$, symmetric in the sense of (23).

Proof: If the calculus is universal, then $\pi_n = \mathrm{id}_{\Omega^n}$ and therefore $\sigma = -\mathrm{id}$. From (23) follows that $g = -g$, hence $g \equiv 0$.

Such consequence is rather an undesired one, as one of its aftermath would be the elimination Higgs-field components of the Standard Model Lagrangian, as we shall see later. Therefore, we should rather stick to the basic interpretation of the symmetry property (22) of the metric.

4.4 Metric on higher order forms

Another standard property of the metric is the possibility of extending its definition for modules of higher-order forms. We shall propose here a scheme for generalisation of it in noncommutative geometry. First, we shall extend g to the tensor products of Ω^1

Observation 2

The metric g extends to $\Omega^{\otimes_{\mathcal{A}} n} \otimes_{\mathcal{A}} \Omega^{\otimes_{\mathcal{A}} n}$ in the following way:

$$\begin{aligned} g(u_1 \otimes_{\mathcal{A}} \ldots \otimes_{\mathcal{A}} u_n \otimes_{\mathcal{A}} v_1 \otimes_{\mathcal{A}} \ldots \otimes_{\mathcal{A}} v_n) \; &= \\ = \; g\left(u_1 \left(\ldots g(u_{n-1} g(u_n \otimes_{\mathcal{A}} v_1) \otimes_{\mathcal{A}} v_2\right) \otimes_{\mathcal{A}} \ldots \otimes_{\mathcal{A}} v_n\right), \end{aligned} \tag{32}$$

which satisfies the basic requirements (21-22).

Now, using the result (16) we may extend the metric for higher order forms using the embedding ρ. For instance, in the case of two-forms this would be:

$$g(\omega, \eta) \; = \; g\left(\rho(\omega), \rho(\eta)\right). \tag{33}$$

for any two-forms ω, η.

4.4.1 Example - metric on two-forms

Here we shall demonstrate how the metric acts on an arbitrary two-form of our product geometry $\mathcal{F}$:

$$\mathcal{F} \;=\; dx^i \wedge dx^j F_{ij} + dx^i \wedge \chi \Psi_i + \chi \wedge \chi \Phi, \tag{34}$$

Using the form of the metric (24-27) and the definition (33) we find:

$$\begin{aligned} g(\mathcal{F}^\star, \mathcal{F}) \;=\; & - \; F^\star_{ij} F_{kl} g^{ik} g^{kl} && (35)\\ & + \; \frac{1}{2} \Psi^\star_i \Psi_j g(\mathcal{R}(g^{ij}) + g^{ij}) && (36)\\ & - \; \Phi^\star \Phi g \mathcal{R}(g). && (37) \end{aligned}$$

The first term is a standard one, coming only from the part of continuous geometry, whereas the last comes only from the discrete geometry and the middle one is mixed. Had we assumed the symmetry condition (23) to hold, we would have had consequently $g = 0$ and both additional terms that have origins in discrete geometry would not appear. This would have profound consequences for physics, as any field theory, and gauge theory in particular, would not feel the presence of discrete geometry (apart from the simple fact that we would have had two seperate copies of each field). In such situation no Higgs-type model would be possible to obtain from the noncommutative geometry based on the product of $\mathbb{R}^n \times \mathbb{Z}_2$, and one should look for models, which involve products of differential calculi, which are not universal, to obtain nontrivial results.

5 Linear Connections

As the standard methods of differential geometry use rather the language of vector fields than differential forms, the translation of the concept of linear connection is a delicate problem. We might also look at the formulation of gauge theory in noncommutative geometry to guess the best definition, let us remind that for any left-module $\mathcal{M}$ over $\mathcal{A}$ the covariant derivative D is defined as a map $\mathcal{M} \to \Omega^1 \otimes_{\mathcal{A}} \mathcal{M}$ such that:

$$D(am) = da \otimes_{\mathcal{A}} m + aD(m), \tag{38}$$

which could be then extended to the degree 1 operator $D : \Omega \otimes_{\mathcal{A}} \mathcal{M} \to \Omega \otimes_{\mathcal{A}} \mathcal{M}$.

One could easily apply this definition for the case of linear connections (and the related covariant derivative) by replacing $\mathcal{M}$ with the appropriate object in this case, the bimodule Ω^1. The problem starts when we begin to look at the bimodule structure of Ω^1 and ask how D acts on ua, $u \in \Omega^1$ and $a \in \mathcal{A}$. Of course, this action is determined by the bimodule structure on Ω^1 and the definition (38), however, it remains to be said whether some extra conditions should be assumed. The only limitation is that the introduced additional restrictions should reduce to (38) in the case of commutative differential calculus.

5.1 Bimodule linear connection

The proposition that one should use the bimodule isomorphism σ to define such property, has been put forward by Dubois-Violette, Madore and others [9-11]:

$$D(\omega a) \;=\; \sigma(\omega \otimes_{\mathcal{A}} da) + D(\omega)a. \tag{39}$$

Throughout this paper we shall call connections that use (in any form) the bimodule property of Ω^1 *bimodule connections*. Indeed, this reduces to the standard expression in the classical case, where $\omega a = a\omega$ and, since $\sigma(\omega \otimes_{\mathcal{A}} \rho) = \rho \otimes_{\mathcal{A}} \omega$ we have (39) equivalent to (38). We shall see, however that this condition is very restrictive in noncommutative case, in fact, as shown recently [9], in many cases of noncommutative Kaluza-Klein theories thet only existing bimodule linear connections have no mixed terms. We shall discuss it later while applying the theory to our example of $\mathbb{R}^n \times \mathbb{Z}_2$.

5.2 Torsion and curvature

Let us observe that (38) is, as mentioned earlier, easily extendible to $\Omega\otimes_{\mathcal{A}}\Omega^1$ according to the rule:

$$D(u \otimes_{\mathcal{A}} \rho) \;=\; du \otimes_{\mathcal{A}} \rho + (-1)^{\deg u} u \wedge D(\rho), \tag{40}$$

where $u \in \Omega$ and $\rho \in \Omega^1$. We can calculate then the curvature D^2 and show that it is left-linear:

$$D^2(u \otimes_{\mathcal{A}} \rho) \;=\; u \wedge D(\rho). \tag{41}$$

Similar extension is not possible for the right-multiplication property of the covariant derivative, and, what is more important one cannot assure that the curvature D^2 is right-linear.

The torsion could be defined as the following map $T : \Omega \otimes_{\mathcal{A}} \Omega^1 \to \Omega$:

$$T = \pi \circ D - d \circ \pi, \tag{42}$$

where π is standard projection. From the construction it is clear that T is a left-module morphism (in case of symmetric connections it is a bimodule homomorphism).
Finally, let us make some general observations on linear connections in non-commutative geometry, which later would be useful.

Observation 3

If D and D' are two linear connections, then $D - D'$ is a left-linear morphism $\Omega \otimes_{\mathcal{A}} \Omega^1 \to \Omega \otimes_{\mathcal{A}} \Omega^1$ of grade 1, moreover, if they are bimodule connections (i.e. obeying (39)) then $D - D'$ is a bimodule morphism.

If Ω^1 is a free bimodule and $\omega_1, \ldots, \omega_n$ form its base, then a connection D such that $D(\omega_i) = 0$ is called *trivial* in this base. Then, as a result of observation 3 we may observe that in that case every connection is a sum of this trivial connection and a left-module (or bimodule in the case of bimodule connections) morphism of grade 1.
To end this section we shall observe that having a $\star$-structure on our external algebra we cannot easily relate somehow $D(u)$ with $D(u^\star)$. However, let us observe that in the classical differential geometry it is not true that $D(\omega^\star) = D(\omega)^\star$

5.3 Ω^1 as a bimodule over Ω

As we have shown in the previous paragraph, the use of bimodule properties of linear connection is rather complicated. In what follows we should attempt to propose a solution, which would make both the notation and results simpler. The price we have to pay is the introduction of additional structure on our differential algebra, as we shall assume that there exist a bimodule structure over Ω (treated as an algebra) on Ω^1. We shall call this bimodule $\mathcal{M}$, assuming that the following conditions hold:
1. $\mathcal{M}$ is generated by elements of the form $u \otimes_{\mathcal{A}} \omega$ where $u \in \Omega$ and $\omega \in \Omega^1$. Of course, $\Omega^1 \subset \mathcal{M}$
2. The left- and right-multiplications by the elements of Ω coincide with $\otimes_{\mathcal{A}}$ if the element of the module is in Ω^1 and $\wedge$ otherwise.

3. $\pi : \mathcal{M} \to \Omega$ defined on the generators $\pi(u \otimes_{\mathcal{A}} \omega) = u \wedge \omega$ is bimodule morphism
3. There exists a $\star$-operation on $\mathcal{M}$.
We shall demonstrate that such structure exists in the standard differential geometry as well as in few examples of noncommutative geometry. First, let us notice that having defined this structure we could immediately write both rules for D, now seen as a map $D : \mathcal{M} \to \mathcal{M}$ of degree 1:

$$D(u \wedge m) = du \wedge m + (-1)^{\deg u} u \wedge D(m) \tag{43}$$
$$D(m \wedge u) = D(m) \wedge u + (-1)^{\deg m} m \wedge du \tag{44}$$

for $m \in \mathcal{M}$ and $u \in \Omega$.
Now, D^2 is automatically a bimodule morphism!.

5.3.1 Examples

First, we shall demonstrate that this structure exists in the standard commutative differential calculus. Define the right action of Ω on the generators of $\mathcal{M}$ as follows:

$$\omega \otimes_{\mathcal{A}} u = (-1)^{\deg u} u \otimes_{\mathcal{A}} \omega, \tag{45}$$

then this gives a proper bimodule structure on $\mathcal{M}$ and π is a bimodule morphism. We can see that in this case (43) is equivalent to (44), as one would expect.
Now let us turn to noncommutative geometry. For universal calculus one can always introduce the bimodule $\mathcal{M}$ as $\wedge$ is just $\otimes_{\mathcal{A}}$ and $\mathcal{M}$ could be identified with the tensor algebra of differential forms itself. For the simplest possible case of two-point geometry we have:

$$D(a\chi) = \chi(\partial a) \otimes_{\mathcal{A}} \chi + aD(\chi), \tag{46}$$
$$D(\chi a) = D(\chi)a - \chi \otimes_{\mathcal{A}} \chi(\partial a), \tag{47}$$

and we could verify that they agree with each other provided that $D(\chi) = 2\chi \otimes_{\mathcal{A}} \chi$, so that D coincides with d. This result is what we could have expected, observe that since π is just identity map, every torsion-free connection on universal calculus must coincide with d.
Next we shall discuss the product of continuous and discrete geometries with the following construction of $\mathcal{M}$. The bimodule structure on $\mathcal{M}$ is, for products of the forms dx^i, just as in the case of continuous geometry,

as discussed above. Similarly, for products of χ alone, we take it as in the example of universal calculus right above. What we have to add is the rule of right multiplication between dx^i and χ:

$$dx^i \otimes_{\mathcal{A}} \chi \quad \sim \quad -\chi \otimes_{\mathcal{A}} dx^i \tag{48}$$

$$\chi \otimes_{\mathcal{A}} dx^i \quad \sim \quad -dx^i \otimes_{\mathcal{A}} \chi \tag{49}$$

We could verify now what (43-44) imply on the covariant derivative. First let us compare $D(adx^i)$ with $D(dx^i a)$:

$$D(adx^i) = dx^j \otimes_{\mathcal{A}} dx^i(\partial_j a) + \chi \otimes_{\mathcal{A}} dx^i(\partial a) + aD(dx^i), \tag{50}$$

$$D(dx^i a) = D(dx^i)a + dx^j \otimes_{\mathcal{A}} dx^i(\partial_j a) + \chi \otimes_{\mathcal{A}} dx^i(\partial a), \tag{51}$$

where in the second relation we have used the rules of right multiplication on $\mathcal{M}$. By comparing the right-hand side of these relations we immediately get that:

$$D(dx^i) \; = \; \Gamma^i_{jk} dx^j \otimes_{\mathcal{A}} dx^k + \alpha\chi \otimes_{\mathcal{A}} \chi. \tag{52}$$

The other pair of relations is:

$$D(a\chi) = dx^i \otimes_{\mathcal{A}} \chi\mathcal{R}(\partial_i a) - \chi \otimes_{\mathcal{A}} \chi(\partial a) + aD(\chi), \tag{53}$$

$$D(\chi\mathcal{R}(a)) = D(\chi)\mathcal{R}(a) + dx^i \otimes_{\mathcal{A}} \chi\mathcal{R}(\partial_i a) + \chi \otimes_{\mathcal{A}} \chi(\partial a), \tag{54}$$

and by comparing the right-hand sides we get;

$$D(\chi) \; = \; 2\chi \otimes_{\mathcal{A}} \chi. \tag{55}$$

Suppose now that we demand that this connection has a vanishing torsion (42). We have already observed that (55), which is equal to the connection on $\mathbb{Z}_2$ alone is torsion-free. For (52) the vanishing of torsion is equivalent to $\alpha = 0$ and $\Gamma^i_{jk} = \Gamma^i_{kj}$, so in the end we obtain that the linear connection on $\mathbb{R}^n \times \mathbb{Z}_2$ splits into separate components, each operating on one element of the product. Therefore, the curvature has also such property, additionally, as in our case D on the discrete space is flat $D^2 = 0$, we have the resulting total curvature operator to have only the standard contribution coming from the continuous element of the product. This would suggest that already on this level, without even introducing the concept of metric connection, we are certain that for bimodule linear connections there would be no modifications to gravity, coming from effects of noncommutative geometry on $\mathbb{R}^n \times \mathbb{Z}_2$. We shall see, that if we drop the requirement of bimodule property (in either form) we can proceed with the construction, which shall lead to some interesting and unexpected features.

6 Metric linear connections

In this sections we shall discuss the generalisation of the idea of metric connections. The form of the definition depends on our assumptions concerning the bimodule properties of D - as our main task is to apply the theory to the considered example and we have already shown that for bimodule connections give no new features in the theory, we shall concentrate on connection, which only satisfy (38) alone.

We say that D is *metric* if the following holds for all one-forms u, v:

$$dg(u, v^\star) \;=\; g(D(u), v^\star) - g(u, D(v)^\star), \tag{56}$$

where we use a shorthand notation: $g(u_1 \otimes_{\mathcal{A}} u_2, v) = u_1 g(u_2, v)$. This definition is well-defined for any D and it gives precise prescription for metric connection in the commutative limit.

The most general form of torsion-free D is:

$$\begin{aligned} D(dx^\mu) &= \Gamma^\mu_{\nu\rho} dx^\nu \otimes dx^\rho + \Gamma^\mu_\nu \left(dx^\nu \otimes \chi + \chi \otimes dx^\nu \right), && (57) \\ D(\chi) &= 2\chi \otimes \chi + B_{\mu\nu} dx^\mu \otimes dx^\nu + W_\nu \left(dx^\nu \otimes \chi + \chi \otimes dx^\nu \right). && (58) \end{aligned}$$

where $\Gamma^\mu_{\nu\rho} = \Gamma^\mu_{\rho\nu}$ and $B_{\mu\nu} = B_{\nu\mu}$. Using the metric (24-27) and the definition (56) we end up with the following set of relations:

$$\begin{aligned} \partial_\rho g^{\mu\nu} &= \Gamma^\mu_{\rho\kappa} g^{\kappa\nu} + \Gamma^\nu_{\rho\kappa} g^{\mu\kappa}, && (59) \\ \partial g^{\mu\nu} &= \mathcal{R}(\Gamma^\mu_\kappa) g^{\kappa\nu} - \mathcal{R}(g^{\mu\kappa}) \Gamma^\nu_\kappa, && (60) \\ \Gamma^\mu_\nu g &= g^{\mu\kappa} B_{\nu\kappa}, && (61) \\ 0 &= g^{\mu\nu} W_\nu, && (62) \\ \partial_\mu g &= 2 W_\mu g, && (63) \\ \partial g &= 0 && (64) \end{aligned}$$

and, as we assume that $g^{\mu\nu}$ is non-degenerate, we immediately get that $W_\mu = 0$ and $g = const$. This simplifies the curvature $R = D^2$ and e have:

$$\begin{aligned} R(dx^\mu) \;=\; & dx^\alpha \wedge dx^\beta \;\left(\partial_\alpha \Gamma^\mu_{\beta\gamma} - \partial_\beta \Gamma^\mu_{\alpha\gamma} + \Gamma^\mu_{\alpha\kappa} \Gamma^\kappa_{\beta\gamma} - \Gamma^\mu_{\beta\kappa} \Gamma^\kappa_{\alpha\gamma} - \Gamma^\mu_\alpha B_{\beta\gamma} + \Gamma^\mu_\beta B_{\alpha\gamma} \right) \otimes dx^\gamma \\ & + \; dx^\alpha \wedge \chi \;\left(-\partial \Gamma^\mu_{\alpha\gamma} - \mathcal{R}(\Gamma^\mu_{\alpha\kappa} \Gamma^\kappa_\gamma) + \partial_\alpha \mathcal{R}(\Gamma^\mu_\gamma) + \mathcal{R}(\Gamma^\mu_\kappa) \Gamma^\kappa_{\alpha\gamma} \right) \otimes dx^\gamma \\ & + \; \chi \wedge \chi \;\left(2\Gamma^\mu_\gamma - \partial \Gamma^\mu_\gamma - \Gamma^\mu_\kappa \mathcal{R}(\Gamma^\kappa_\gamma) \right) \otimes dx^\gamma \\ & + \; dx^\alpha \wedge dx^\beta \;\left(\partial_\alpha \Gamma^\mu_\beta - \partial_\beta \Gamma^\mu_\alpha + \Gamma^\mu_{\beta\kappa} \Gamma^\kappa_\alpha - \Gamma^\mu_{\alpha\kappa} \Gamma^\kappa_\beta \right) \otimes \chi \end{aligned}$$

$$
\begin{aligned}
& + \; dx^\alpha \wedge \chi \;\; (-\partial\Gamma^\mu_\alpha - 2\mathcal{R}(\Gamma^\mu_\alpha) + \mathcal{R}(\Gamma^\mu_\kappa)\Gamma^\kappa_\alpha) \otimes \chi
\end{aligned}
$$

$$
\begin{aligned}
R(\chi) \;\; &= \;\; dx^\alpha \wedge dx^\beta \;\; \Big(\partial_\alpha B_{\beta\gamma} - \partial_\beta B_{\alpha\gamma} + \Gamma^\kappa_{\alpha\gamma} B_{\beta\kappa} - \Gamma^\kappa_{\beta\gamma} B_{\alpha\kappa}\Big) \otimes dx^\gamma \\
& + \; dx^\alpha \wedge \chi \;\; \Big(2B_{\alpha\gamma} - \partial B_{\alpha\gamma} - \mathcal{R}(B_{\alpha\kappa}\Gamma^\kappa_\gamma)\Big) \otimes dx^\gamma \\
& + \; dx^\alpha \wedge dx^\beta \;\; \Big(\Gamma^\kappa_\alpha B_{\beta\kappa} - \Gamma^\kappa_\beta B_{\alpha\kappa}\Big) \otimes \chi
\end{aligned} \tag{65}
$$

Now, if we use (61), we may eliminate $B_{\mu\nu}$ from the expressions for R. Furthermore, it will be convenient to use θ^μ_ν: $\Gamma^\mu_\nu = \delta^\mu_\nu + \theta^\mu_\nu$. First, we may rewrite (60):

$$
\mathcal{R}(\theta^\mu_\kappa) g^{\kappa\nu} \;=\; \mathcal{R}(g^{\mu\kappa})\theta^{\nu\kappa}_{,} \tag{66}
$$

or using the inverse $g_{\mu\nu}$:

$$
\mathcal{R}(\theta^\kappa_\mu g_{\nu\kappa}) \;=\; \theta^\kappa_\nu g_{\kappa\mu}. \tag{67}
$$

The curvature tensor, rewritten using only using $\Gamma^\mu_{\nu\rho}$ and θ^μ_ν variables (only in the first line we still use Γ^μ_ν) is:

$$
\begin{aligned}
R(dx^\mu) \;\; &= \;\; dx^\alpha \wedge dx^\beta \;\; \Big(\partial_\alpha \Gamma^\mu_{\beta\gamma} - \partial_\beta \Gamma^\mu_{\alpha\gamma} + \Gamma^\mu_{\alpha\kappa}\Gamma^\kappa_{\beta\gamma} - \Gamma^\mu_{\beta\kappa}\Gamma^\kappa_{\alpha\gamma} - g g_{\nu\gamma}\Big(\Gamma^\mu_\alpha \Gamma^\nu_\beta - \Gamma^\mu_\beta \Gamma^\nu_\alpha\Big)\Big) \otimes dx^\gamma \\
& + \; dx^\alpha \wedge \chi \;\; \Big(\mathcal{R}(\theta^\mu_\kappa)\Gamma^\kappa_{\alpha\gamma} - \mathcal{R}(\Gamma^\mu_{\alpha\kappa}\theta^\kappa_\gamma) + \partial_\alpha \mathcal{R}(\theta^\mu_\gamma)\Big) \otimes dx^\gamma \\
& + \; \chi \wedge \chi \;\; \Big(\delta^\mu_\gamma - \theta^\mu_\kappa \mathcal{R}(\theta^\kappa_\gamma)\Big) \otimes dx^\gamma \\
& + \; dx^\alpha \wedge dx^\beta \;\; \Big(\partial_\alpha \theta^\mu_\beta - \partial_\beta \theta^\mu_\alpha + \Gamma^\mu_{\beta\kappa}\theta^\kappa_\alpha - \Gamma^\mu_{\alpha\kappa}\theta^\kappa_\beta\Big) \otimes \chi \\
& + \; dx^\alpha \wedge \chi \;\; (-\delta^\mu_\alpha + \mathcal{R}(\theta^\mu_\kappa)\theta^\kappa_\alpha) \otimes \chi
\end{aligned}
$$

$$
\begin{aligned}
R(\chi) \;\; &= \;\; dx^\alpha \wedge dx^\beta \;\; (g g_{\mu\gamma}) \Big(\partial_\alpha \theta^\mu_\beta - \partial_\beta \theta^\mu_\alpha + \Gamma^\mu_{\beta\kappa}\theta^\kappa_\alpha - \Gamma^\mu_{\alpha\kappa}\theta^\kappa_\beta\Big) \otimes dx^\gamma \\
& + \; dx^\alpha \wedge \chi \;\; (g g_{\mu\gamma}) \, (\delta^\mu_\alpha - \theta^\mu_\kappa \mathcal{R}(\theta^\kappa_\alpha)) \otimes dx^\gamma \\
& + \; dx^\alpha \wedge dx^\beta \;\; (g g_{\kappa\rho}) \Big(\theta^\kappa_\alpha \theta^\rho_\beta - \theta^\kappa_\beta \theta^\rho_\alpha\Big) \otimes \chi
\end{aligned}
$$

and we see that some expressions repeat itself in the structure of the curvature tensor. The Ricci tensor R_c is the trace of the curvature tensor:

$$
R_c \;\; = \;\; dx^\beta \otimes dx^\gamma \;\; \Big(\partial_\mu \Gamma^\mu_{\beta\gamma} - \partial_\beta \Gamma^\mu_{\mu\gamma} + \Gamma^\mu_{\mu\kappa}\Gamma^\kappa_{\beta\gamma} - \Gamma^\mu_{\beta\kappa}\Gamma^\kappa_{\mu\gamma} - g g_{\nu\gamma}\Big(\Gamma^\mu_\mu \Gamma^\nu_\beta - \Gamma^\mu_\beta \Gamma^\nu_\mu\Big)\Big)
$$

$$
\begin{aligned}
&- \; dx^\beta \otimes dx^\gamma \;\; \frac{1}{2}(g g_{\mu\gamma}) \left(\delta^\mu_\beta - \theta^\mu_\kappa \mathcal{R}(\theta^\kappa_\beta)\right) \\
&+ \; \chi \otimes dx^\gamma \;\; \frac{1}{2} \left(\mathcal{R}(\theta^\mu_\kappa)\Gamma^\kappa_{\mu\gamma} - \mathcal{R}(\Gamma^\mu_{\mu\kappa}\theta^\kappa_\gamma) + \partial_\mu \mathcal{R}(\theta^\mu_\gamma\right) \\
&+ \; dx^\beta \otimes \chi \;\; \mathcal{R}\left(\partial_\mu \theta^\mu_\beta - \partial_\beta \theta^\mu_\mu + \Gamma^\mu_{\beta\kappa}\theta^\kappa_\mu - \Gamma^\mu_{\mu\kappa}\theta^\kappa_\beta\right) \\
&+ \; \chi \otimes \chi \;\; \frac{1}{2}\mathcal{R}\left(-\delta^\mu_\mu + \mathcal{R}(\theta^\mu_\kappa)\theta^\kappa_\mu\right)
\end{aligned} \tag{68}
$$

and finally the curvature scalar, which is simply the value of the metric on the Ricci tensor:

$$
R \;=\; g^{\beta\gamma}\left(\partial_\mu \Gamma^\mu_{\beta\gamma} - \partial_\beta \Gamma^\mu_{\mu\gamma} + \Gamma^\mu_{\mu\kappa}\Gamma^\kappa_{\beta\gamma} - \Gamma^\mu_{\beta\kappa}\Gamma^\kappa_{\mu\gamma}\right) - g\left(\Gamma^\mu_\mu \Gamma^\beta_\beta - \Gamma^\mu_\beta \Gamma^\beta_\mu\right), \tag{69}
$$

Such result is an interesting one - we shall get the action, which is a sum of two standard Hilbert-Einstein actions for gravity (one for $g\mu\nu(x,+)$ and the other one for $g^{\mu\nu}(x,-)$) as well as additional terms, which depend only on both metric tensors (no derivatives!) and the field θ^κ_β (satisfying (67)). This would suggest that such term plays the role of a constraint, which enforces relations between $g^{\mu\nu}(x,+)$ and $g^{\mu\nu}(x,-)$. In the simplest possible case, when the are equal to each other, it would reduce itself to the cosmological constant term. This would recover the results obtained by [13, 14] using different approach based on the Dirac operator and Wodzicki residue. Further and more detailed discussion on the properties of the obtained model of gravity on $\mathbb{R}^n \times \mathbb{Z}_2$ and example solutions shall be presented elsewhere.

7 Conclusions

In this paper we have presented few schemes, which have been considered as a generalisation of linear connections (and related objects) in noncommutative geometry. Our main aim was to apply these methods to a simple example of noncommutative Kaluza-Klein type model, being the product of continuous ($\mathbb{R}^n$) and discrete ($\mathbb{Z}_2$ geometries. Our choice has been motivated by the interpretation of the electroweak part of the Standard Model, in which such geometry plays an important role providing the explanation of the origin of Higgs field.

We have found that most concepts ale easily translated from standard differential geometry to the noncommutative case and give reasonable results in our example. However, some others, especially the postulate of symmetry

imposed on the metric and bimodule properties of linear connections, can cause rather significant problems. In particular, in our example each of these requirements has profound consequences. In the first case, it eliminates all discrete degrees of freedom in the field theory, whereas in the second case, it gives no new features of gravity in this setup. Though the latter may be considered as an acceptable result, we cannot agree with the former - as, we already know how the theory should look like [1, 15].
Therefore we definitely cannot accept the generalisation of symmetric metric as discussed here (we still require that is hermitian), being aware that the other result might also suggest the second postulate (bilinear linear connections) goes too far. In our considerations we have also proposed another version of this postulate, which makes it more natural. One of its main advantages is that R becomes a bimodule morphism.
On the other hand we have provided a derivation of gravity-type theory for our example of product geometry, based on the assumption that only left-linearity is important for linear connections, obtaining quite a feasible result.

Of course, it still remains open, whether the accepted methods are proper for noncommutative geometry, as they are based on what we have learned from standard differential geometry. The main problem is that few features, which coincide in the commutative case, are different if we turn on noncommutativity. One has to choose, which property is appropriate in such situation and different choices may give completely different results. It is also not clear why the standard methods must be followed in noncommutative case, for instance, we might ask why we have to set the torsion to zero.
We have demonstrated in this paper some good points and problems of two methods as applied to a simply and - realistic - model. The results that we have found are important for determination of some fundamental concepts of noncommutative geometry, however, they have to be verified using other methods, so that they could be accepted or properly generalised for noncommutative geometry, which remains a big task for future research.

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